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# A Study on Correlated Exponential Random Walks

**T. M. John[1] and K. P. N. Murthy[1]**

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Based on the linear Boltzmann transport formulation, we investigate the statistics of correlated exponential random walks that are continuous in space and discrete in time. We show that asymptotically, the correlated random walk process is diffusive and derive an effective diffusion constant. We investigate the power spectral characteristics of the associated random forces. We also present some results on the first passage time distribution and establish that asymptotically it reduces to that associated with simple Gaussian walks.

**KEY WORDS:** Random walks; first passage time problems; exponential random walks; correlated random walks; telegraph equation; power spectral density; linear Boltzmann transport equation.

## 1. INTRODUCTION

Recently, there have been numerous studies on the correlated random walk (CRW) and its characteristics.[1–6] These walks were first introduced by Taylor[7] and extensively investigated by Goldstein.[8] The CRW can be considered as the simplest example of a multistate random walk.[9] For a review and references to some early work on this subject, see Ref. 10. The interest in CRW stems from the fact that it provides useful models in the study of several diffusion problems in physics, chemistry, biology, and sociology.[11–20] Also, recently these walks have found some interesting applications in problems on neutron transport in media with anisotropic scatterers.[21–24]

The characteristics of CRW that are of interest are the conditional spatial probability distribution, the first passage time distribution,[26,27] and the power spectrum of the associated noise sources.[4] The asymptotic form

[1] Radiation Shielding and Statistical Physics Section, Reactor Physics Division, Indira Gandhi Centre for Atomic Research, Kalpakkam, 603 102, Tamil Nadu, India.



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of the spatial probability distribution of the CRW has been shown to be Gaussian.[1-8] Specifically, the interest is in the asymptotic nature of the variance of the spatial distribution.[1,3,4,10,27]

The general procedure adopted in the studies of CRW consists in considering random walks on a one-dimensional lattice with nearest neighbor jumps. The random walk starts at the origin and jumps to the left (right) site with probability $\alpha_0$ ($\beta_0 = 1 - \alpha_0$). In the subsequent steps one defines $\alpha$ as the probability for the random walk to persist in its direction and $\beta$ ($= 1 - \alpha$) as the probability to reverse its direction. A set of recursion relations is written for the process and solved under a suitable continuum limit.

It is known that the diffusion equation describes the asymptotic behavior of continuous space random walks generated by arbitrary jump density with finite second moment and is also the continuum limit of simple lattice walks.[28] It is not clear, however, whether one can write down directly a differential equation for describing the CRW generated by a Gaussian jump density. The continuum limit of the discrete recursion relations established for the CRW on a lattice can be obtained (see, for example, Ref. 4). Indeed, it has been shown (see, for example, Ref. 10) that in the continuum limit of the lattice and time, the CRW is described by the diffusion equation (for arbitrary $\alpha$) and by the telegrapher's equation in the limit $\alpha \to 1$.

It is the purpose of this paper to show that a set of difference-differential equations can be obtained for describing discrete time and continuous space correlated walks generated by exponential jump density, by making a suitable analogy to the linear Boltzmann transport equation. The derivation of the equations is accomplished by considering balance conditions for the random walks entering, exiting, and getting removed in an infinitesimal line segment. The details of the derivation are given in Section 2, where we also show that in the continuous time limit the set of equations reduce to the telegrapher's equation. In Section 3, we present some numerical results on the first passage time distribution and derive analytic expressions for describing its asymptotic behavior. We show that in the asymptotic region, the first passage time distribution of the CRW process reduces to that associated with Gaussian random walks. We present in Section 4 results on the power spectrum of the stationary noise that drives the correlated random walk process. Our results also show the phase transition-like phenomenon reported in Ref. 4. But we find the process to be diffusive, asymptotically for all values of $\alpha$, unlike the collapsed (for $\alpha = 0$) or deterministic (for $\alpha = 1$) behavior reported in Ref. 4. The principal conclusions of the study are brought out in Section 5.

## 2. MATHEMATICAL FORMULATION

In this section, we derive from first principles a coupled set of difference-differential equations for describing the spatial probability distribution of the correlated exponential random walk process in one dimension. To this end we define $P^{R}(x, n)\,dx$ and $P^{L}(x, n)\,dx$ as the probability of the random walk to be at $x$ within $dx$ after $n$ number of jumps and moving to the right and left, respectively. The probability for the random walk starting off from a point to visit a site at a distance $x$ within $dx$ in a single step is given by a negative exponential density,

$$f(x) = \lambda^{-1} \exp(-x/\lambda) \tag{1}$$

where $\lambda$ is a known parameter. Having jumped to the site, the walk can persist in its direction with probability $\alpha$ or reverse its direction with probability $\beta = 1 - \alpha$.

Next we consider an infinitesimal line segment $\Delta x$ at $x$, and set up a balance equation for the loss and gain of $n$-step, right-directed random walks. The loss terms arise due to (1) net transport, given by

$$P^{R}(x + \Delta x, n) - P^{R}(x, n) = \frac{\partial P^{R}(x, n)}{\partial x} \Delta x \tag{2}$$

and (2) the termination at $x$ within $\Delta x$, which, by virtue of the exponential jump density, is given by $\lambda^{-1} P^{R}(x, n)\,\Delta x$. The gain term comes from the right- and left-directed random walks after the $(n-1)$th step at $x$ within $\Delta x$, and persisting and reversing the direction. The gain term is thus given by

$$\lambda^{-1}[\alpha P^{R}(x, n-1) + \beta P^{L}(x, n-1)]\,\Delta x \tag{3}$$

Thus, balancing the gain and the loss terms, we get for the right-directed $n$-step walks a recursion relation given by

$$\frac{\partial P^{R}(x, n)}{\partial x} + \lambda^{-1} P^{R}(x, n) = \lambda^{-1}[\alpha P^{R}(x, n-1) + \beta P^{L}(x, n-1)] \tag{4}$$

Similarly, for the $n$-step left-moving random walks, we get

$$-\frac{\partial P^{L}(x, n)}{\partial x} + \lambda^{-1} P^{L}(x, n) = \lambda^{-1}[\alpha P^{L}(x, n-1) + \beta P^{R}(x, n-1)] \tag{5}$$

The coupled recursion relations (4) and (5) are for $n>1$. For $n=1$, however, we have the source conditions given by

$$\frac{\partial P^{\mathrm{R}}(x, n=1)}{\partial x}+\lambda^{-1} P^{\mathrm{R}}(x, n=1)=S^{\mathrm{R}}(x) \tag{6}$$

$$-\frac{\partial P^{\mathrm{L}}(x, n=1)}{\partial x}+\lambda^{-1} P^{\mathrm{L}}(x, n=1)=S^{\mathrm{L}}(x) \tag{7}$$

We choose the source distribution as

$$S^{\mathrm{R}}(x)=\delta(x)/2 \tag{8}$$

$$S^{\mathrm{L}}(x)=\delta(x)/2 \tag{9}$$

The above choice of source distribution implies that the first step is symmetric in its direction.

Let us now define

$$P(x, n)=P^{\mathrm{R}}(x, n)+P^{\mathrm{L}}(x, n) \tag{10}$$

$$J(x, n)=P^{\mathrm{R}}(x, n)-P^{\mathrm{L}}(x, n) \tag{11}$$

Then we get

$$\begin{aligned} \frac{\partial J(x, n)}{\partial x}+\lambda^{-1} P(x, n) &= \lambda^{-1} P(x, n-1), \qquad n>1 \\ &= \delta(x), \qquad n=1 \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{\partial P(x, n)}{\partial x}+\lambda^{-1} J(x, n) &= \lambda^{-1}(2\alpha-1)\, J(x, n-1), \qquad n>1 \\ &= 0, \qquad n=1 \end{aligned} \tag{13}$$

The above equations can be easily reduced by eliminating $J(x, n)$ to yield

$$\begin{aligned} \frac{\partial^2 P(x, n)}{\partial x^2} =\; & 2\lambda^{-2}(1-\alpha)[P(x, n)-P(x, n-1)] \\ & + \lambda^{-2}(2\alpha-1)[P(x, n)-2P(x, n-1)+P(x, n-2)], \quad n>2 \end{aligned} \tag{14}$$

$$\frac{\partial^2 P(x, n)}{\partial x^2}=\lambda^{-2} P(x, n)+2\lambda^{-2}\alpha P(x, n-1)+\lambda^{-1}(2\alpha-1)\, \delta(x), \quad n=2 \tag{15}$$

$$\frac{\partial^2 P(x, n)}{\partial x^2}=\lambda^{-2} P(x, n)-\lambda^{-1} \delta(x), \quad n=1 \tag{16}$$

In order to obtain a continuum limit of Eq. (14), we let $n\tau = t$ and Taylor expand $P(x, t-\tau)$ and $P(x, t-2\tau)$ up to the second order in $\tau$. We then get

$$\frac{\partial^2 P(x,t)}{\partial x^2} = \frac{2(1-\alpha)\tau}{\lambda^2}\frac{\partial P(x,t)}{\partial t} + \frac{\alpha\tau^2}{\lambda^2}\frac{\partial^2 P(x,t)}{\partial t^2} \tag{17}$$

We next consider the matter of scaling. Two cases emerge. In the first case we take the limits $\lambda$ and $\tau \to 0$ such that

$$\lim_{\lambda \to 0, \tau \to 0} \frac{\lambda^2}{\tau} = D_0 \tag{18a}$$

When the above substitution is made in Eq. (14) we get the diffusion equation

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} \tag{18b}$$

with diffusion constants $D = D_0/2\beta$. In the second case we take the limit $\lambda$, $\tau$, and $\beta$ $(=1-\alpha) \to 0$, such that

$$\lim_{\lambda \to 0, \tau \to 0, \beta \to 0} \begin{cases} \lambda/\tau = V \\ \tau/\beta = T \end{cases} \tag{18c}$$

Then we get

$$\frac{\partial^2 P(x,t)}{\partial x^2} = \frac{2}{V^2 T}\frac{\partial P(x,t)}{\partial t} + \frac{1}{V^2}\frac{\partial^2 P}{\partial t^2} \tag{18d}$$

The above is the telegrapher's equation (see, e.g., Ref. 10), where the constants $V$ and $T$ have the dimensions of velocity and time, respectively.

However, if we set $\alpha = \frac{1}{2}$ in Eq. (14), we recover, under the continuum limit, the diffusion equation with a diffusion constant $D_0$.

## 3. FIRST PASSAGE TIME DISTRIBUTION

In this section we consider the CRW on the real line extending from $-\infty$ to $L > 0$. The walk starts at the origin and its initial direction is symmetrically distributed. We define a random variable $N$ that denotes the number of steps the walk takes to cross the barrier for the first time. We want to obtain $\Pi(N)$, the probability distribution of the first passage time $N$.

Formally, the first passage time distribution can be written as

$$\Pi(N) = \int_{-\infty}^{L} \lambda^{-1}[P(x, N-1) - P(x, N)]\, dx \tag{19}$$

where $P(x, N)$ is the solution of Eq. (14) with suitable boundary condition. Using Eq. (12) in the above, we can write

$$\Pi(N) = \int_{-\infty}^{L} \frac{\partial J(x, N)}{\partial x}\, dx = J(x = L, N) \tag{20}$$

The appropriate boundary condition for this problem is

$$P^{L}(x = L, N) = 0 \qquad \text{for all } N \tag{21}$$

which implies

$$P(x = L, N) = J(x = L, N) \tag{22}$$

Thus, for determining $\Pi(N)$ we need only to solve Eqs. (14)–(16) with the required boundary conditions and get $P(x = L, N)$. To this end, we introduce a discrete transform of variable $N$ to $S$ as

$$\tilde{P}(x, s) = \sum_{n=1}^{\infty} \exp[-S(n-1)]\, P(x, n) \tag{23}$$

Using the above in Eq. (14)–(16), we get

$$\frac{\partial^2 \tilde{P}(x, s)}{\partial x^2} = \lambda^{-2} a(s)\, b(s)\, \tilde{P}(x, s) - \lambda^{-1} b(s)\, \delta(x) \tag{24}$$

where

$$a(S) = 1 - e^{-S} \tag{25}$$

$$b(S) = 1 - (2\alpha - 1)\, e^{-S} \tag{26}$$

Equation (24) can be readily solved, and using Eqs. (20) and (22), we get

$$\tilde{\Pi}(S) = \{1 + [a(S)/b(S)]^{1/2}\}^{-1} \exp\{-(L/\lambda)[a(S)\, b(S)]^{1/2}\} \tag{27}$$

Formally, $\tilde{\Pi}(S)$ can be expanded in powers of $e^{-S}$ and the coefficient of $e^{-NS}$ would yield the required $\Pi(N)$. This has been carried out employing a simple numerical procedure and the results are shown in Fig. 1 for the values of $\alpha = 0.0$, 1/3, 1/2, and 2/3 and $\lambda = 1$. A closed-form expression for the asymptotic behavior of $\Pi(N)$ can, however, be obtained as follows. We

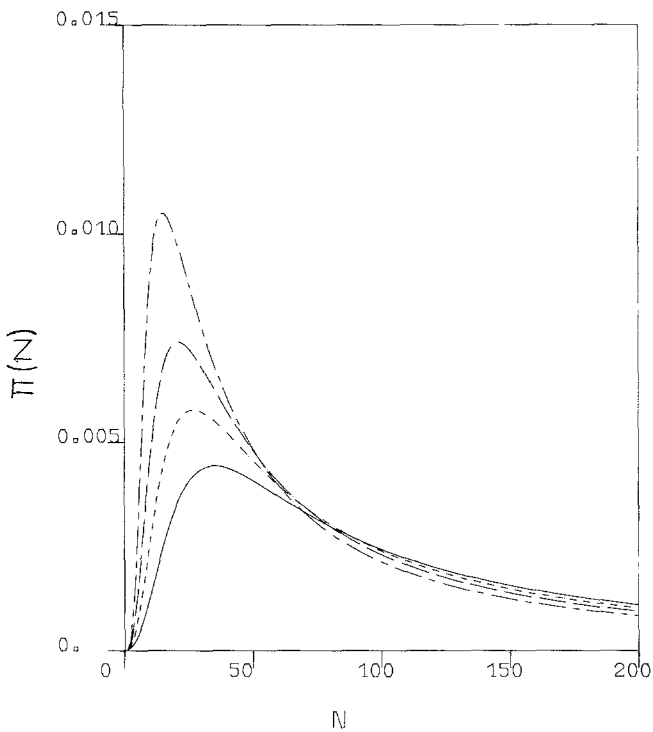


Fig. 1. Plot of the first passage time distribution for various values of $\alpha$: (—) 0.0, (- -) 1/3, (— —) 1/2, and (—-—) 2/3.

let $S \to 0$ in Eq. (24) (keeping in mind that $\alpha$ is not close to unity) and obtain

$$\operatorname*{Lt}_{S \to 0} \tilde{\Pi}(S) = \exp[-L(S/D)^{1/2}] \tag{28}$$

The Laplace inverse of the above expression is

$$\Pi_A(T) = L[T(4\Pi DT)^{1/2}]^{-1} \exp(-L^2/4DT) \tag{29}$$

The above expression is clearly the first passage time associated with simple Gaussian walks (see, e.g., Ref. 30).

Setting, $S = 0$ in Eq. (27), we find that $\tilde{\Pi}(S = 0) = 1$. This implies that the first passage time distribution $\Pi(N)$ is properly normalized, and this establishes the "gambler's ruin" for such correlated games.

## 4. SPECTRAL CHARACTERISTICS OF DRIVING NOISE

In this section we investigate the power spectral characteristics of the noise that drives the CRW. To motivate this, let us consider a process dictated by the Langevin equation:

$$dx(t)/dt = \eta(t) \tag{30}$$

where $\eta(t)$ is the driving noise. If $\eta(t)$ is stationary, Gaussian, and white, then $x(t)$ is the Weiner process, which describes the continuum of Gaussian walks. Thus the Langevin equation provides an alternative to the diffusion equation. The Langevin description has its own advantages, especially for problems where we are interested only in the first few moments of the process.

In the same spirit we wish to obtain the corresponding Langevin model for the CRW process. In other words, we wish to characterize the color of the stationary noise $\eta(t)$ in Eq. (30) that would render $x(t)$ a correlated process. It is clear that the power spectrum of $\eta(t)$ would depend on the parameter $\alpha$ and we would like to characterize this dependence.

The power spectral density of $\eta(t)$ is given by (see Ref. 4)

$$\phi(\omega) = -\tfrac{1}{2}\omega^2 \mathscr{L}[\langle x^2(t)\rangle] = -\tfrac{1}{2}\omega^2 \tilde{M}_2(i\omega) \tag{31}$$

where $\mathscr{L}$ is the Fourier–Laplace transform operator.

In our problem, since we are concerned with discrete time, we employ the discrete transform defined by Eq. (23) with $S = iw$, and formally express the power spectrum of driving noise as

$$\phi(w) = [1 - \exp(-iw)]^2\, \tilde{M}_2(iw) \tag{32}$$

where $\tilde{M}_2(iw)$ is obtained by using Eq. (24) and is

$$\tilde{M}_2(iw) = \frac{2}{\lambda^2}[1 - \exp(-iw)]^{-2}[1 - (2\alpha - 1)\exp(-iw)]^{-1} \tag{33}$$

Inserting the above in Eq. (32), we get

$$\begin{aligned} \phi(w) = {} & (1/2\lambda^2)[\beta^2 + (2\alpha - 1)\sin^2(w/2)]^{-1} \\ & \times \{[\beta + (2\alpha - 1)\sin^2(w/2)] + (i/2)(2\alpha - 1)\sin w\} \end{aligned} \tag{34}$$

The real and imaginary parts of the power spectrum are depicted in Figs. 2 and 3, respectively, for the cases with $\alpha = 0$, 1/3, 1/2, and 2/3. For

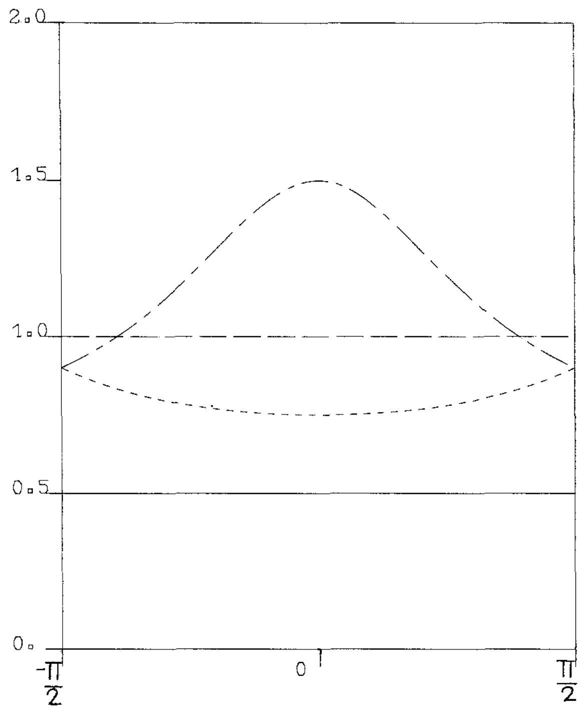


Fig. 2. Plot of the real part of the power spectrum versus frequency for various values of $\alpha$: (—) 0.0, (- -) 1/3,(— —) 1/2, and (—-—) 2/3.

$\alpha = 1/2$ we recover the white spectrum, as expected. The results depicted in Figs. 2 and 3 agree qualitatively with those reported in Ref. 4. However, we get a white spectrum even for $\alpha = 0$, arising due to diffusion associated with the randomness in the step size of the random walk. The power spectrum shows distinct features in two phases, a "collapsed phase" and "counter diffusive phase" characterized by $\alpha < 0.5$ and $\alpha > 0.5$, respectively, thus exhibiting phase transition-like behavior around the isotropic critical point $\alpha = 0.5$. In the limit of $w \to 0$, $\phi(w)$ reduces to that reported in Ref. 4 except for a small deviation arising due to the difference in the effective diffusion constant. This phase breaking is a consequence of the anisotropy. This picture, in addition to being a model for anisotropic neutron transport, as shown in this paper, could also be useful to study several different physical systems that can be described by correlated processes.

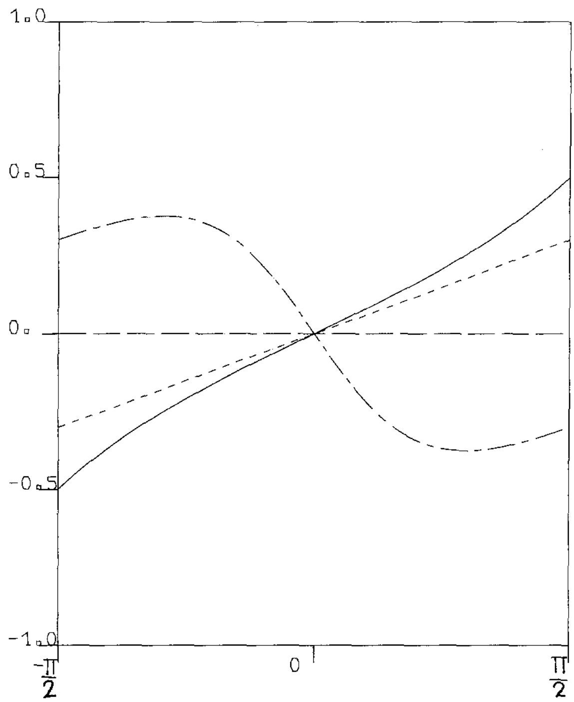


Fig. 3. Plot of the imaginary part of the power spectrum versus frequency for various values of $\alpha$: (—) 0.0, (- -) 1/3, (— —) 1/2, and (—-—) 2/3.

## 6. CONCLUSIONS

The main aim of this paper has been to present a formulation that interprets the stationary linear Boltzmann transport equation as providing a natural description for discrete time, continuous space exponential walks.

We have applied this formulation to investigate three important characteristics of correlated random walks. We show that the process is diffusive asymptotically and derive an expression for the effective diffusion constant. Our results on the spectral characteristics of the noise that drives the CRW agree qualitatively with those reported earlier.[4] We have also presented some results on the first passage time distribution and shown that it reduces asymptotically to that associated with simple Gaussian walks.

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